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ASTRONOMY, ASTROPHYSICS, AND COSMOLOGY

The Entropy of a Gravitating Body

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Abstract—This paper proposes a generalization of the dependence for the entropy of "ordinary" massive bodies with a relatively small entropy of the event horizon of the covering surface to the case of black holes (BH). By doing this, the nature of the famous Bekenstein bound, that is, the universal limit for entropy, is immediately explained and corrections to the values of the "gravitational" surface temperature are determined for the entire spectrum of astrophysical objects.

Keywords: black hole, entropy, temperature, density, thermodynamics, surface gravity, gravitational collapse, Bekenstein limit, Hawking temperature, Unruh temperature, action at a distance.

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INTRODUCTION

In the early 1970s, a group of relativistic cosmologists made a revolutionary breakthrough in understanding BH thermodynamics [1]. Based on Hawking's result on the nondecreasing absolute event horizon, Bekenstein [2] identified the BH entropy with the surface area of this horizon. Even earlier, first Zeldovich [3], and much later, Hawking himself [4, 5], contrary to his initial ideas, showed that a BH should radiate as a black body with an appropriate temperature. The rest of the cosmological community accepted these ideas gradually and with great doubt.

According to Bekenstein [6, 7], the enormous entropy of a BH occurs because the state of a BH does not provide information about the system, by which growth it was formed. The BH entropy is considered the maximum possible value of entropy for objects of a given mass, that is, the Bekenstein bound, although this statement has never been rigorously proved.

Physicists usually perceive the thermodynamics of BHs with the colossal entropy inherent to them as something exotic that has no analogy with the thermodynamics of ordinary objects. In this paper, we show that a "smooth" transition from gravitationally conditioned thermodynamics of "ordinary objects" to black holes is possible, and substantiate the famous Bekenstein bound: it does actually give an upper bound for the entropy of a gravitating object.

The entropy of an ordinary macroscopic body in many cases can be found from the thermodynamic formulation, when macroscopic rather than microscopic state parameters are used [8–11]. This is the reason for state uncertainty. The specific microstate of the body is unknown, and it does not always exist for a long period of time.

The interactions between mutually distant parts of a macroscopic system, for example, molecules in a drop of water or in a vessel filled with gas, can often be discarded, because intermolecular forces are short-range in nature. The entropy of such systems consists of the entropy of all its subsystems, in other words, it is additive in volume, that is, an extensive quantity. In particular, let us split a body in equilibrium into macroscopic subsystems for which the equilibrium temperature has the same value T. Heat energy E_{macro} is

$$E_{\text{macro}} = T \sum_V \Delta S_i = T S_{\text{macro}},$$

where ΔS_i is the *volume* entropy of the ith subsystem and S_{macro} is the entropy of a macroscopic body, which is additive in volume.

However, not all systems can be described by conventional Boltzmann thermodynamics, for example, a sufficiently large interstellar dust cloud or a similar large cosmological object. If we split the cloud into imaginary parts, then these parts will interact not along interfaces, but via entire volumes, that is, each part will "sense" the entire system as a whole, all its other parts. For this reason, for example, thermodynamic additivity is badly violated in gravitating systems: such a system cannot be divided into

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approximately independent subsystems. Entropy in such systems is not additive in volume.

Nevertheless, the sources of the gravitational field have similar properties to the above-described "macroscopic" systems. In particular, they can be described by choosing macroscopic parameters similar to the concepts of temperature and entropy. Moreover, these new thermodynamic parameters effectively translate into parameters used in the limiting case for gravitational objects such as BHs.

So, what are the nature and properties of the entropy of gravitating objects? Verlinde suggested that a relationship between gravity and entropy may exist for "ordinary" massive bodies far from gravitational collapse, and not only for BHs [12]. For each such body, he introduced the concept of a hypothetical spherical screen that surrounds this body and carries information about it and suggested that the forces of gravity are not fundamental, but secondary in nature, determined precisely by the magnitude of the entropy gradient that arise when the radius of the hypothetical screen changes. As a result, Verlinde concluded that the entropy S and the area A of the sphere bounding the body are directly proportional.

However, the assumption about the presence of such a "virtual" screen carrying information does not seem to be justified (see, in particular, the interesting article [13]). On the other hand, the undoubted connection between gravity and entropy is not necessarily due to the primary nature of entropy. For example, the authors of [14] show that another point of view is valid: the thermodynamic state equation is derived from the equations of the gravitational theory.

1. THE ENTROPY OF AN OBJECT WITH GRAVITATIONAL INTERACTION BETWEEN ITS PARTS

We contrast the Verlinde hypothesis with the opposite statement: it is not entropic forces that are primary, but gravitational forces, and the gravitational field should create conditions for the arising of entropy gradients not only in the case of BHs, but also in the case of "ordinary" massive bodies.

In 1687 Isaac Newton published the popular "shell theorem" [15], according to which "a spherically symmetric body affects external objects gravitationally as if all its mass were concentrated at a point at its center." This causes macroscopic uncertainty: the gravitational force acting on the test body at a distance r from the center of mass of a homogeneous and symmetric source of gravitation does not depend on how the mass M of the source is distributed within an imaginary sphere of radius r. (It is clear that "ordinary" bodies rarely possess perfect symmetry and homogeneity. Deviations from homogeneity and symmetry can lead to a shift in the position of the center of mass and "deformation" of closed equipotential surfaces of the gravitational field compared to the spherical case, but these effects can usually be discarded at distances from the center greater than several diameters of the body.) However, such uncertainty is associated with the appearance of entropy when different configurations of subsystems correspond to the same "macroscopic" description of the system for an external observer.

Let us introduce the main parameters of an object that is a source of gravity with mass M and radius r of an imaginary spherical shell bounding it.

The mean density ρ of matter in the object:

$$\rho = \frac{3M}{4\pi r^3}. \tag{1}$$

The surface gravity κ, i.e., the magnitude of the gravitational acceleration experienced by a small test particle on a conditional outer spherical bounding shell:

$$\kappa = \frac{GM}{r^2} = \frac{4\pi G\rho r^3}{3r^2} = \frac{4\pi G\rho r}{3}, \tag{2}$$

where G is the gravitational constant. Relation (2) defines nothing other than, for example, the acceleration of a particle in the gravitational field of the planet.

The surface temperature (in ergs) $\Theta = k_\mathrm{B}T$ on a certain outer spherical bounding shell, or the total energy of the body, related to the area A of the shell surface:

$$\Theta = k_\mathrm{B}T = \frac{Mc^2}{A} = \frac{Mc^2}{4\pi r^2} = \frac{GM}{r^2}\frac{c^2}{4\pi G} = \kappa\frac{c^2}{4\pi G}. \tag{3}$$

Here, $k_\mathrm{B} = 1.38 \times 10^{-16}$ erg/deg is the Boltzmann constant, and T is temperature in K.

As can be seen from (2) and (3), gravity and surface temperature are proportional to each other. Since the Planck length $l_p = 1.6 \times 10^{33}$ cm can be written as:

$$l_p = \sqrt{\hbar G/c^3},$$

where $\hbar$ is the reduced Planck's constant, then relation (3) can be reduced to the form that will be used further:

$$T = \kappa\frac{c^2}{4\pi G k_\mathrm{B}} = \kappa\frac{\hbar c^2/c^3}{4\pi(\hbar G/c^3)k_\mathrm{B}} = \frac{\hbar\kappa}{\pi c k_\mathrm{B}}\frac{1}{4l_p^2}. \tag{4}$$

The last parameter is the entropy of the gravity source. When analyzing it, it is important to keep in mind the following circumstance. Just as for a "macroscopic" system in equilibrium, we assumed the volume temperature T to be the same for all its subsystems, in this case, based on the shell theorem, we assumed its surface analogue T to be the same for each area of the surface A of the sphere. At first glance, this thesis seems trivial. However, in this regard, it is appropriate to recall the "hoop conjecture" [1], a definite alternative to the Newtonian shell theorem: the imploding object forms a BH if and only if this object can be entirely placed in the critical sphere, when it is possible to "place a hoop" around the object. Consequently, in the case of an insufficiently compact and symmetric object, a state with the same surface temperature may not exist. However, if it is possible to surround the object with a sphere, then, according to (3) and the simplified expression

$$dE = \frac{E}{A}dA = TdA,$$

the area of a sphere plays the role of entropy up to a coefficient and the total entropy corresponding to such a sphere is additive with respect to the surface of the sphere, i.e., it is the sum of the values of the entropies of partial areas on the sphere surface.

Finally, following Bekenstein's example, we consider the fall of a test particle on a gravitational field source. We suppose that a test particle crosses an imaginary sphere of a certain radius r, surrounding the source, which is not a BH in this case. For another test particle outside this sphere, it looks as if the mass of the source has increased due to the first test mass and, accordingly, the number of possible configurations of the mass distribution inside this sphere has increased. That is, the first test mass brings the entropy associated with it into the sphere, which exactly resembles the situation with a black hole.

2. BH AND THE BEKENSTEIN BOUND

Thus, as noted in Section 1, in the first half of the 1970s, a chain of events took place that led to an understanding of BH thermodynamics, in particular to Zeldovich's and later Hawking's idea of BH radiation as if its event horizon had a finite temperature proportional to the BH surface gravity. Bekenstein established that a BH should have enormous entropy S_0 proportional to the area A_0 of its bounding sphere, that is, the event horizon [2]:

$$S_0 = \frac{A_0}{4l_p^2}.$$

We present the expressions for ρ_0, κ_0, and $\Theta_0 = k_{\mathrm{B}}T_0$, written precisely for a BH of mass M, the gravitational Schwarzschild radius of which, as we know, is equal to [16]

$$R_0 = \frac{2MG}{c^2}.$$

The critical density ρ_0 of matter in the collapsed object:

$$\rho_0 = \frac{3M}{4\pi R_0^3}. \tag{5}$$

The surface gravity κ_0, i.e., the magnitude of the gravitational acceleration at the BH event horizon is

$$\kappa_0 = \frac{GM}{R_0^2} = \frac{4\pi G\rho_0 R_0}{3}. \tag{6}$$

The surface temperature $\Theta_0 = k_{\mathrm{B}}T_0$ at the BH event horizon:

$$\Theta_0 = k_{\mathrm{B}}T_0 = \kappa_0\frac{c^2}{4\pi G}. \tag{7}$$

Comparing formulas (1)–(3) and (5)–(7), we see that

$$\frac{\rho}{\rho_0} = \frac{R_0^3}{r^3},$$
$$\frac{\kappa}{\kappa_0} = \frac{T}{T_0} = \frac{R_0^2}{r^2}.$$

We note that formula (2) implies the existence of acceleration κ for a test particle in the gravitational field of an arbitrary object. Due to the equivalence principle, this immediately implies the Unruh effect [17], that is, the existence of thermal radiation with the corresponding temperature T (see (4)) in the accelerating frame of reference, and the acceleration κ itself is identified with the Unruh acceleration.

Although the Unruh effect is usually associated with the quantum theory of a vacuum, our case did not require it. It is sufficient to account for the simple fact that an accelerated charge in the space surrounding the source of gravity always radiates, drawing energy from the gravitational field. Moreover, at the BH event horizon the acceleration κ and the Unruh temperature T become the acceleration κ_0 and the Zeldovich–Hawking temperature T_0.

3. UNIFYING DESCRIPTIONS OF AN "ORDINARY" BODY AND A BH

Above, we obtained for the entropy of an arbitrary gravitation source the same proportionality to its surface area, and not to volume, as for a BH. A similar result was obtained by Verlinde for the entropy of the holographic horizon, but this led to a fundamental paradox, which Verlinde himself noted: if the proportionality coefficient for an "ordinary body" is taken the same, then the entropy of a BH turns out to be much less than the entropy of an ordinary body, since the

Table 1. The ρ/ρ_0 ratio for various astrophysical objects

Object	Mass M, kg	Radius r, m	Gravitational radius r_0, m	$(\rho/\rho_0) = (R_0/r)^3$	$(\kappa/\kappa_0) = (\Theta/\Theta_0) = (R_0/r)^2$
Earth	6×10^{24}	6×10^6	10^{-2}	$\sim 10^{-26}$	$\sim 10^{-18}$
Sun	2×10^{30}	7×10^8	3×10^3	$\sim 10^{-16}$	$\sim 10^{-10}$
Milky Way	3×10^{42}	$\sim 10^{19}$	$\sim 10^{15}$	$\sim 10^{-12}$	$\sim 10^{-8}$
Universe	$\sim 10^{53}$	$\sim 10^{26}$	$\sim 10^{26}$	~ 1	~ 1

gravitational radius of this body for a given mass M until collapse is much less than the actual one!

What causes this paradox? The fact is that we actually used a relation of the following form when calculating the temperature of the gravitational source

$$\frac{1}{\Theta} = \left(\frac{\partial S}{\partial E}\right)_V,$$

where the derivative of entropy S with respect to energy E is calculated at constant volume V [18]. However, we are considering the process of approaching collapse of a body of a given mass M, characterized by a changing density $\rho = M/V$, and this change cannot be considered small. In this case, we must also consider the change in volume in the expression for temperature Θ. This can be done by introducing a certain factor depending on the value (ρ/ρ_0). Expanding this factor in a series in powers of this value, we restrict ourselves to the first-order term; in fact, we equate this factor to this value itself. This clearly does not change anything for a BH, but for an "ordinary" body, for example, an astrophysical object, it reduces this entropy by many orders of magnitude (see Table 1).

We also note that usually the factor $1/4l_p^2$ is also "transferred" from the expression for temperature to the expression for the surface area A of the bounding sphere, thus making the entropy dimensionless. Therefore, we propose the following final formula for the gravitational entropy

$$S \approx \frac{\rho}{\rho_0}\frac{A}{4l_p^2}.$$

The approximation sign occurs due to our restricting ourselves by the first order of the power expansion.

CONCLUSIONS

Let us sum up this work; for any massive particle, an entropy gradient arises naturally, directed along the radius of the gravitational field. In this sense, the connection between the "fundamental" force and entropy is not at all a privilege of gravitational interaction, therefore, Verlinde's concept seems untenable.

The unification of the entropy expression for ordinary objects and BH using formula (3) is based on the linear approximation of an unknown exact dependence on (ρ/ρ_0) and makes it possible to describe both the "ordinary" states of massive bodies with a relatively small entropy of the imaginary horizon of the bounding surface, and the state of BH with huge entropy on the event horizon. By doing this, the nature of the famous Bekenstein limit, the universal limit for entropy, is immediately explained and substantiated, and corrections to the values of the "gravitational" surface temperature are determined for the entire spectrum of astrophysical objects.

REFERENCES

1. K. S. Thorn, *Black Holes and Time Warps: Einstein's Outrageous Legacy* (W. W. Norton, New York, 1994), p. 619.
2. J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
3. B. Ya. Zel'dovich, Sov. Phys. JETP **35**, 1085 (1972).
4. J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973); S. W. Hawking, Nature (London, U.K.) **248**, 30 (1974).
5. S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
6. J. D. Bekenstein, Contemp. Phys. **45**, 31 (2003).
7. J. D. Bekenstein, Lett. Nuovo Cim. **4**, 737 (1972).
8. J. W. Gibbs, *Elementary Principles of Statistical Mechanics* (Ox Bow Press, Woodbridge, 1981).
9. Ya. M. Gelfer, *History and Methodology of Thermodynamics and Statistical Physics,* 2nd ed. (Vysshaya Shkola, Moscow, 1981) [in Russian].
10. Ya. M. Gelfer, V. L. Lyuboshits, and M. I. Podgoretskii, *Gibbs Paradox and Identity of Particles in Quantum Mechanics* (Nauka, Moscow, 1975), p. 272 [in Russian].

11. D. V. Sivukhin, *General Course of Physics* (Nauka, Moscow, 1990), Vol. 2, p. 591 [in Russian].
12. E. P. Verlinde, High Energ. Phys., No. 04, 029 (2011). https://doi.org/10.1007/JHEP04(2011)029
13. Yun Soo Myung, Hyung Won Lee, and Yong-Wan Kim, "Entropic force versus temperature force," arXiv:1006.1922v1 [hep-th] (2010).
14. F. Porcelli and G. Scibona, "On the black hole's thermodynamics and the entropic origin of gravity," arXiv: 1011.0895.
15. I. Newton, *Philosophiae Naturalis Principia Mathematica* (Imprimatur, S. Pepys, London, 1686), Theorem XXXI.
16. I. D. Novikov and V. P. Frolov, *Physics of Black Holes* (Nauka, Moscow, 1986), p. 328 [in Russian].
17. W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
18. L. D. Landau and E. M. Lifshits, *Course of Theoretical Physics,* Vol. 5: *Statistical Physics* (Nauka, Moscow, 1976; Pergamon, Oxford, 1980).

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