© M.H. Shulman, 2008 (shulman@dol.ru)

How to transit to quantum mechanics from classical one
(Updated 02.11.2008)

I argue that the transition to the quantum mechanics from the classical one should include two important steps. Firstly, a usual "slow" motion described by the standard dynamics laws and by real quantities should be added by a quickly oscillating motion component and the complex quantities usage. Secondly, for this motion fast component the postulate should be accepted that all these oscillators are nonlocal, extend over the all Universe and take in account the current Universe size.

Quantum mechanics and complex values

As it is known, quantum mechanics (QM) appeared at the first third of 20th century as result of classical science crisis. After that QM achieved a great success, but is not free from some fundamental vagueness.

At 1925 W. Heisenberg had firstly represented a quantum particle position and velocity as an infinite serie of complex-valued harmonics (see [**Jammer, 1967]**). At the same time, he limited its amplitudes and multiplication rules (which become similar to the matrix multiplication rules), so all the following was deduced from these conditions.

Because of that the complex-valued functions appeared in QM, however, as it turned out, its physical meaning was not clear. M. Born at 1926 proposed the statistical interpretation of wave function, that allowed to calculate the quantum processes probability distributions.

The domain of complex values applications in quantum mechanics is developing up to now. Particularly, many publications use the transition to imaginary time (see, for example, the review **[Vainshtein et al., 1982]**); a number from them (e.g., **[Wu, 1995])** try to base by this way the results of the experiments **[Chiao et al., 1995]** relative to photon traveling through a potential barrier. The other authors work with a complex particle trajectory: for example, in **[Poirier, 2008]** the probability density behavior along a complex-valued trajectory is studied, and author concludes that it does not conserve; however, the author of **[John, 2008]** considers the probability density behavior along a *real axis* of a complex particle trajectory, and this allows to introduce the extended Born's probability axiom, while the density can be found from the imaginary part of the particle velocity field.

In my opinion, the more general approach provides to include the *complex* energy, coordinate and velocity into the classical mechanics, that allows describe a particle behavior in a different classically forbidden regions as well qualitatively as quantitatively. A classical particle behavior having real or complex energy is compared in **[Bender et al., 2008]**. On the basis of extensive numerical studies the authors argued that there are strong analogies between the probabilistic behavior of quantum systems defined by Hermitian Hamiltonians and the deterministic behavior of classical (one-dimensional) mechanical systems extended into the complex domain (a particle position is also represented by a complex value[1]).

In cited work is noted that a wave packet that is initially localized in one potential well can tunnel to the other one. Complex solutions to the classical equations of motion exhibit a remarkably analogous behavior. Furthermore, classical solutions come in two varieties,

---

[1] To run ahead, I would like to note that (as I will show below) imaginary energy component addition is physically equivalently to addition some quickly oscillating particle coordinate and velocity components, however, authors of the paper **[Bender et al., 2008]** do not mention about it.

which resemble the even-parity and odd-parity quantum-mechanical bound states. For the periodic potential a quantum particle can tunnel repeatedly between adjacent classically allowed regions and thus execute a localized random walk as it hops from region to region. Furthermore, if the energy of the quantum particle lies in a conduction band, then the particle delocalizes and drifts freely through the periodic potential. A classical particle having complex energy executes a qualitatively analogous local random walk, and there exists a narrow energy band for which the classical particle becomes delocalized and moves freely through the potential.

Thereupon, the next important questions may be posed:

- What a physical meaning do complex values have in classical mechanics?
- What (besides complex values) do we need add to classical mechanics in order to do it equivalent enough to QM?

These questions were in detail studied in the book **[Shulman, 2004]**, here my point of view is given more compactly.

Analogy between QM and alternating current circuits theory

Generally, the complex quantities are used since 17th century to describe physical phenomena, so there are enough the precedents. However, one remarkable analogy exists that allows to understand the background and the way to apply the quantum mechanical mathematics. It is the analogy with the alternating current circuits theory.

The known specialist in the area of current circuits theory G. Kron proposed to simulate the Shrödinger equation solution using linear circuits **[Kron, 1945]**[2] and found good agreement with analytical one. What about me, I have observed at 2004 the remarkable resemblance between two bilinear expressions: In QM a simplest probability to detect a particle is given as $\Psi^*\Psi$, while in the alternating current circuits theory the branch power is equal to U^*I (U is electrical tension, I is electrical current, the asterisk denotes the complex conjugation operation).

Here we consider a tension U and current I as complex harmonic quantities, i.e. they have some amplitude, frequency, and phase. At the same time, they also may be described by real quantities:

$$u = u_m \sin \omega t, \quad i = i_m \sin(\omega t + \varphi),$$

where t is time, ω is cyclic frequency, φ is the current phase shift relative to tension, index "m" denotes the maximal values. The instantaneous power (which presents the product of u and i) can be transformed to

$$s = (u_m i_m /2) [\cos \varphi - \cos (2\omega t - \varphi)]$$

First term in the square bracket is constant, it is the *active* power that is *irreversibily* dissipated by the branch. Second term pulsates with the frequency 2ω and *mean* value equal to *zero*, it corresponds to a *purely ossilating energy exchange*.

In order to get the *full branch power* W *averaged over period* in the frame of complex representation one usually multiplies either the conjugate complex U* by complex I or, countrary, complex U by conjugate complex I* (to calculate W*). By such way a

[2] P.V. Kurakin informed me about it at 2007.

dependence on time disappears due to addition the terms (iωt) and (-iωt) in the exponent, and we have the complex result

$$W = (u_m\, i_m / 2)\, e^{i\varphi} = (u_m\, i_m / 2)\, (\cos\varphi + i \sin\varphi)$$

Here the *real* part corresponds to the *active* power, while the *imaginary* part corresponds to the *reactive* (oscillating) power.

Note that the reactive aspect account in the classical area may be illustrated using the following example: a *direct* current cannot flow in a branch containing a capacity, however, an *alternate* current can easy "tunnel" through it.

Commutation rules for oscillators

Of course, all this is correct for every physical oscillator, not only for an electrical circuit. Particularly, for a mechanical oscillator we should use complex position Q, complex momentum Q and complex action S instead of tension U, current I and power W respectively. The main feature consist namely in the complex presentation of these quantities. The complex of action (energy, power) play here the role completely analogous of the role of the product Ψ*Ψ in QM.

Moreover, it is easy to build the *commutation rules* for complex values that are similar to these ones for physical quantities in QM. For example, we have for action S of a *one-dimensional* mechanical oscillator expressed through position Q and momentum P

$$[Q, P] \equiv Q^* \cdot P - Q \cdot P^* = S - S^* = q_m\, p_m\, (i \sin\varphi).$$

where φ is a phase shift of P relative to Q. If this shift is equal to the quarter-period, then we have an oscillator without power loss and $[Q, P] = i\, q_m\, p_m$.

In fact, the quantum particles, as it turned out, correspond to *two-dimensional* oscillators without power loss, not to one-dimensional ones. Hense, in such case they have *two degrees of freedom*. If the maximal values q_m , p_m and the oscillation frequency ω remain the same ones, then the new degree of freedom corresponds to the oscillations phase shift between two different spatial axis. So, two-dimensional oscillators may have an additional distinction, one-dimensional oscillators (including electrical circuits) cannot have it. It is very important fact, because it leads to the spin effect.

Let us name as *two-dimensional oscillator of 1-st genus* (pulsator) a such one whose the oscillations along both the spatial axis are in-phase, i.e. the phase shift between them is *zero*. Further, let us name as *two-dimensional oscillator of 2-st genus* (rotator) a such one whose the oscillations between two spatial axis are *shifted on the quarter-period* (either positive or negative). It is enough to consider a positive shift; if it is a negative one, then one can exchange the spatial axis indexes). In my book **[Shulman, 2004]** I showed that the oscillators of 1-st genus satisfy to the commutation rules for bosons, while the oscillators of 2-st genus satisfy to the commutation rules for fermions (the more, they have the *classical* feature of spin).

One can ask, is it possible in the frame of this approach to base the famous *Pauli principle*? I propose to consider two oscillators 2-st genus system as *three-dimensional* oscillator of 2-st genus. In such oscillator *each* pair of oscillations has to be chifted on the quarter-period between two axis, but it is impossible, hense Pauli principle should be complished for fermions. Countrary, for bosons the pulsations in-phase along all three axis can be realized.

Transition to probability, "disappearance" of determinism

We have got the commutation rules whose right side contains an action (energy, power) amplitude multiplied by imaginary unit (энергии, мощности). These relationships for *oscillators of 2-genus* (fermions) describe a balance of reactive component over the oscillation period (mean value is zero). In principle, we can study any small part of this period. However, if we analyze the times that are much larger than this one, then we practically transit to the statistical process consideration.

Traditional QM just uses such statistical approach. The known theorem was formulated by Kennard and Robertson (see [**Jammer, 1967**]) as *statistical* one: *if* each of two quantum operators is specified by a certain stochastic measurement dispersion of the corresponding quantity, *then* the commutator of these operators may be expressed through the square root of these dispersions product (in fact, it is the proof of the Heisenberg uncertainty principle). *However, this theorem may accomplish even if the origin dispersions are deterministic, not stochastic!*

We can conclude that QM explicitly refuses a deterministic description, it uses a probabilistic representation of processes that occur inside of a specific time-spatial intervals corresponding to wave origin of these processes[3].

Nonlocality of QM

The Universe nonlocality phenomena is very important point of transition from classical mechanics to QM, its role increases during last decades. We have now a large number of experimental works relative to EPR, interferometers, quantum teleportation.

Initially, the nonlocality penetrated into QM via Bohr – Sommerfeld condition of the orbit numerical quantification. The similar conditions for a problem like the potential well lead, for example, to the particle tunneling through a barrier possibility. R. Feynman proposed later the alternative formulation of QM that was based directly on a particle "virtual traveling" possibility over all the available for it trajectories in the Universe (see Appendix).

The results coincidence based on such different formulations says that the nonlocality condition is the real meaning of all physical phenomena. However, one has to introduce the same Planck constant ħ in the both version "by hand", its origin is not clear enough[4].

Let us return to the consideration of our oscillators without power loss for which we use a comlex (not real) Hamiltonian

$$H(p, q) = p^2/2m + i\, kq^2/2$$

Note that in this case the Hamilton's function $H(p, q)$ becomes the complex conjugate to the Lagrange's one $L(p, q)$, and the (complex-valued) energy magnitude remains constant due to the power loss absence.

---

[3] As it is shown in the paper **[Wang and Xiong]**, a particle even can propagate over a space-like interval, which is due to the Heisenberg's uncertainty relation and in agreement with quantum field theory.

[4] The authors of the paper **[Bender et al., 2008]** note that the quantification possibility for their classical trajectories is like Bohr-Sommerfeld rule, but they do not write about necessity to select the concrete empirical constant ħ.

When one uses real quantities only, the product of coordinate and momentum is commutative, however, as we saw, their complex representations are not. In classical mechanics the known relationship is accomplished for Poisson's bracket

$$\{p, q\} = 1$$

In QM the coordinate and momentum commutator is equal to the Poisson's bracket multiplied by ħ and dividing by imaginary unit, where ħ is the elementary action. If one analogously defined a Poisson's bracket via a classical oscillator without power loss action's commutator [P, Q] that was introduced above and is purely imaginary, then he deduced just classical relationship for the Poisson's bracket.

As we can see, the difference is following: For a classical oscillator its (constant) action is expressed via its individual specific parameters q_m and p_m (maximal values of coordinate and momentum), and in QM this action is always equal to the universal Planck constant ħ. Because of that we may conclude that ALL quantum oscillators are specified by the same product $q_m p_m$ and this fact has to have a clear physical explanation.

Let me remember the QM problem of a potential well having finite width. For such well the product of its momentum and de Broglie wavelength is *always* $2\pi\hbar$. The more, let us suggest that the Universe perimeter is finite. If we will increase the well's width up to this perimeter, we will conserve this product value. In my opinion, the single reasonable explanation of this fact consists in that the *momentum is symply proportional to the wavefunction periods number* along the Universe perimeter (the famous Bohr rule). This fact should be true for *each* quantum oscillator!

Conclusion

So, we constructed the model basing on classical mechanics, which represents such important QM features as complex-valued quantities usage (without any mysticism) and transparent commutation rules for two object classes (bosons and fermions). The absolute value of a wave function square that presents in QM the probability to find a particle, in our interpretation has the very close status: it is proportional to the energy density of this particle.

At the same time, we complicated the based concept of dynamics by adding a quickly oscillating (without power loss) component to an "usual" slow law motion of a particle. Furthermore, now the *deterministic* motion laws (like Newton's law) become usable for the slow component only, while on uses, in fact, the *stochastic* approach to describe high frequency oscillation. This approach means some *averaging* over the oscillation period and practically ignores an evolution of motion energy reactive part. Only the active energy dissipation remains to be considered[5].

---

[5] Note that in the Dirac's relativistic equation a free electron corresponds with the velocity eigenvalue, which is *always* equal to the velocity of light. This is linked with so-called "zitterbewegung", i.e. a spiral or circular particle motion that leads to the spin effect. Such motion existence was firstly postulated by Shrödinger at 1930. In order to exclude the "zitterbewegung" from a description one has to eliminate the matrix terms due to transitions between electron and positron states. It is equivalently to the averaging over a period that is much larger than ($\hbar/2mc^2$), where m is the electron mass, c is the velocity of light, ħ is the Planck constant. In the QM non-relativistic problems the oscillations frequencies are determined by energy of the stationary states and transitions between them.

As it was shown in **[Beniaminov, 2007]**, a motion in the compete *classical mechanics* configuration space (coordinates plus momentum), presented by a sum of the "slow" motion and the fast oscillation one

Additionally, we come to the understanding that physical value of the Planck constant has to be determined by the current Universe size.

In my opinion, if one take in account all the above, then he realizes, in generally, the logical transition to QM departing from the classical one.

Appendix

R. Feynman proposed the alternative QM formulation that was directly based on a virtual particle traveling possibility just over all the paths in the Universe. He states the following group of rules:

- A particle movement via single possible trajectory between the initial and final points is corresponding to a *partial* complex-valued probability *amplitude* A ~ exp(– *i*S/ħ), where *i* is imaginary unit, S is particle action for the path interval, ħ is Planck constant.
- *The full* complex-valued probability *amplitude* of the particle displacement between the initial and final points presents the *sum of all the partial amplitudes* over possible pats connecting these two points.
- *The real probability of the particle displacement* between the initial and final points is equal to the modulus square of the complex-valued full probability amplitude.

Here the factor exp(–*i*S/ħ) for any trajectory corresponds with a periodical process having an argument ωt = S/ħ, where ω is the cyclic frequency, and t is time. As the process is periodical, then the phase difference between the initial and final points only is important. In other words, the exponential factor rotates a partial amplitude on certain angle.

Something very similar occurs in an electrical circuit containing an ideal alternating tension that is loaded by a number of parallel branches with inductances and capacitors. Such branch may be specified by its comlex-valued admittance that obtain a phase shift of this branche relative the tension phase. In this case *the branch currents will be summarized (and interfere) just like probability amplitudes in the Feynman rule*.

The more, the total power of all the branches after averaging over the period may be presented as total current multiplyed by the complex conjugate quantity. So, the total power is similar (proportional) to the probability of a particle transition from the initial point to the final one.

I would like to stress here especially that the Feynman's way to transit to QM using the relationship (S/ħ) is linked with the explicit introducing of some *oscillalatory process* like the ideal external alternating tension source action in the elecrical analogy. Further, such oscillatory process is specified by the universal normalizing factor ħ. As it was noted above, this one can be considered as nonlocality factor taking in account the Universe current size.

(where oscillation frequency is $2mc^2/\hbar$), comes to the reduced motion due a specific diffusion process. The fast motion reduces an arbitrary wave function to a function from a subspace whose elements are parameterized by complex-valued functions of coordinates *only*. The slow motion occurs in this subspace and is described by the Schrödinger equation. By the way, this leads a new possible interpretation of the Heisenberg uncertainty principle.

References:

[Bender et al., 2008] Carl M Bender, Dorje C Brody, and Daniel W Hook**.** Quantum effects in classical systems having complex energy. arXiv:0804.4169v1 [hep-th] 25 Apr 2008.

[Beniaminov, 2007] Evgeny Beniaminov. Diffusion Processes in Phase Spaces and Quantum Mechanics. ArXiv:0803.2669v1 [math-ph] 18 Mar 2008.

[Chiao et al., 1995] Raymond Y. Chiao, Paul G. Kwiat and Aephraim M. Steinberg. Quantum Nonlocality in Two-Photon Experiments at Berkeley. arXiv:quant-ph/9501016v1 18 Jan 1995.

[**Jammer, 1967**] Jammer M. The conceptual development of quantum mechanics. Mc Graw-Hill, 1967.

[John, 2008] Moncy V. John. Probability and complex quantum trajectories. arXiv:0809.5101v1 [quant-ph] 30 Sep 2008

[Kron, 1945] Gabriel Kron. Electric Circuit Models of the Schrodinger Equation. Ph. Rev., vol. 67, №№ 1 and 2, January 1 and 5, 1945

[Poirier, 2008] Bill Poirier. On Flux Continuity and Probability Conservation in Complexified Bohmian Mechanics. arXiv:0803.0193v1 [quant-ph] 3 Mar 2008

[Shulman, 2004] Shulman M.H. Variations on Quantum Theory (in Russian). Moscow, Editorial URSS, 2004. The brief English presentation is available at http://timeorigin21.narod.ru/eng_quantum/Eng_variations.pdf

[Vainshtein et al., 1982] Vainshtein A.I., Zakharov V.I, Shifman M. A., Novikov V.A. ABC-book of instanton. Uspekhy Phys. Nauk, 1982, April, vol. 136, № 4, p.p. 553-591

[Wang and Xiong] Zhi-Yong Wang, Cai-Dong Xiong. Quantum-mechanical Lorentz transformation and superluminal phenomenon. http://lanl.arxiv.org/ftp/arxiv/papers/0705/0705.2951.pdf

[Wu, 1995] Wu Zhong Chao. The Imaginary Time in the Tunneling Process. arXiv:0804.0210v1 [quant-ph] 1 Apr 2008.